Combinatorics

Back Paper

Instructions: All questions carry equal marks.

1. Let A be a partial Latin sqaure of order n in which $(i, j)^{th}$ entry is filled if and only if $1 \leq r$ and $j \leq s$. Then prove that A can be completed to a Latin square if and only if
$$N(i) \geq r + s - n, \quad \text{for} \quad 1 \leq i \leq n$$
where $N(i)$ denotes the number of times i occurs in A.

2. Define *mutually orthogonal Latin squares.* Prove that there can be at most $n-1$ mutually orthogonal Latin squares of order n and show that this bound is attained when n is a prime power.

3. Define a t-design. Prove that a t-design is a r-design for all $1 \leq r \leq t$.

4. Prove that in any non-trivial Steiner system $S(t, v, k)$, we must have
$$v \geq (t + 1)(k + 1 - t).$$

5. Prove that a $2-(v, 3, 1)$ design exists if and only if $v \equiv 1$ or $3(\text{mod}6)$.

6. Define *transportation network* and a *cut* and a *flow* in such a network. Prove that in a transportation network the maximum value of a flow equals the minimum value of a cut.

7. Define *combinatorial geometry* and the notion of *independence* in such a geometry. State and prove the semi-modular law in a combinatorial geometry.